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# MAXIMAL ATTRACTOR AND INERTIAL SET FOR EGUCHI-OKI-MATSUMURA EQUATION

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## 1. INTRODUCTION

This is a joint work with Naoto Tanaka (Fukuoka University) and Atsusi Tani (Keio University). We consider following system of equations which was proposed by Eguchi-Oki-Matsumura ([7]):

$$(1.1) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta(-\Delta u + 2u + uv^2), & (x, t) \in Q_T \equiv \Omega \times (0, T), \\ \frac{\partial v}{\partial t} = \beta \Delta v + \alpha v(a^2 - u^2 - b^2 v^2), & (x, t) \in Q_T, \\ \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0, & (x, t) \in \Gamma_T \equiv \Gamma \times (0, T), \\ u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), & x \in \Omega, \end{cases}$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^3$  with smooth boundary  $\partial\Omega \equiv \Gamma$  and  $T > 0$ . Here  $u(x, t)$  is the local concentration of the solute atoms,  $v(x, t)$  is the local degree of order, respectively.  $\alpha, \beta, a, b$  are positive constants and  $\frac{\partial}{\partial n}$  is exterior normal derivative to  $\Gamma$ .

It is well known that phase separation is described by so-called Cahn-Hilliard equation which is fourth order parabolic type [5], while the order-disorder transition is described by Allen-Cahn equation [1]. The system (1.1) is a model of simultaneous order-disorder and phase separation in binary alloys. T. Eguchi, K. Oki and S. Matsumura derived the system (1.1) assuming that the order-disorder transformation is second order and that phase separation can not take place in the disorder state, but can in the ordered state.

In our previous work [11], it was proved that there exist a unique local and global solution to problem (1.1). Many authors studied the dynamics of equations describing phase transition (for example, [2], [3], [4], [10], [12]). In this talk we show the existence of a maximal attractor and of an inertial set to problem (1.1). The main theorems are as follows:

**Theorem 1.** Let  $H_{\bar{u}} = \{(u, v) \in (H^1(\Omega))' \times L^2(\Omega); \frac{1}{|\Omega|} \langle u, 1 \rangle = \bar{u}\}$ . For any  $\delta \geq 0$ , the semigroup  $S(t)$  associated with problem (1.1) possesses in  $\mathcal{H}_\delta = \bigcup_{|\bar{u}| \leq \delta} H_{\bar{u}}$  a maximal attractor  $A_\delta$  that is connected.

**Theorem 2.** Let  $B_\delta$  be the absorbing set in  $(H^1(\Omega))' \times L^2(\Omega)$  and  $X_\delta = \overline{\bigcup_{t \geq t_0} S(t) B_\delta}$ . Then there exists an inertial set  $M_\delta$  for  $(S(t)_{t \geq 0}, X_\delta)$  which has fractal dimension.

## 2. PRELIMINARIES

We shall summarize the results of [11]. First of all the existence theorem is as follows:

**Theorem 3.** For any  $(u_0, v_0) \in (H^2(\Omega))^2$  satisfying the compatibility conditions  $\frac{\partial u_0}{\partial n} \Big|_{\Gamma} = \frac{\partial v_0}{\partial n} \Big|_{\Gamma} = 0$ , problem (1.1) has a unique local solution  $(u, v)$  defined on  $Q_{T'}$  for some  $T' \in (0, T)$  such that

$$(2.1) \quad \begin{aligned} u &\in H^{4,1}(Q_{T'}) \cap C(0, T'; H^2(\Omega)), \\ v &\in L^2(0, T'; H^3(\Omega)) \cap H^1(0, T'; L^2(\Omega)) \cap C(0, T'; H^2(\Omega)). \end{aligned}$$

Here  $H^{4,1}(Q_T) = H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^4(\Omega))$ .

**Theorem 4.** Under the same assumptions of Theorem 3, problem (1.1) admits a unique global solution  $(u, v)$  on  $Q_T$  for any  $T > 0$ .

Problem (1.1) is a gradient flow and it has the Lyapunov functional

$$(2.2) \quad J(u, v) = \int_{\Omega} \left( \frac{1}{2} |\nabla u|^2 + \frac{\beta}{2\alpha} |\nabla v|^2 - \frac{a^2}{2} v^2 + \frac{b^2}{4} v^4 + u^2 + \frac{1}{2} u^2 v^2 \right) dx,$$

which satisfies

$$(2.3) \quad \frac{d}{dt} J(u, v) + \int_{\Omega} (|\nabla K(u, v)|^2 + \frac{1}{\alpha} |v_t|^2) dx = 0,$$

where  $K(u, v) \equiv -\Delta u + 2u + uv^2$ . From (2.3), we have

**Lemma 5.** If  $(u, v)$  satisfies problem (1.1), then

$$(2.4) \quad \begin{aligned} &\frac{1}{2} \|\nabla u\|^2 + \|u\|^2 + \frac{\beta}{2\alpha} \|\nabla v\|^2 + \frac{b^2}{8} \|v\|_{L^4}^4 + \|uv\|^2 \\ &+ \int_0^t ds \int_{\Omega} \left( |\nabla K(u, v)(s)|^2 + \frac{1}{\alpha} |v_t(s)|^2 \right) dx \leq J(u_0, v_0) + \frac{a^4}{2b^2} |\Omega|. \end{aligned}$$

Moreover we can obtain the boundedness of the norm  $\|v(t)\|_{L^\infty}$ .

**Lemma 6.** The estimate

$$(2.5) \quad \sup_{t>0} \|v(t)\|_{L^\infty} \leq C \max \left\{ \|v_0\|_{L^\infty}, \sup_{t>0} \|v(t)\| \right\}$$

is valid for the solution  $(u, v)$  to problem (1.1).

### 3. THE MAXIMAL ATTRACTOR

Let  $H = (H^1(\Omega))' \times L^2(\Omega)$ . We define semigroup  $S(t)$  associated to problem (1.1) by  $(u(t), v(t)) = S(t)(u_0, v_0)$ . Theorems 3 and 4 yield that  $(u, v) = S(\cdot)(u_0, v_0) \in C(0, \infty; H)$ , and that the mapping  $(u_0, v_0) \mapsto (u, v)$  is a continuous operator from  $H$  to  $H$ . For  $u \in (H^1(\Omega))'$  let  $Nu$  be the solution of boundary value problem

$$(3.1) \quad \begin{cases} -\Delta \psi = u - \bar{u}, & x \in \Omega, \\ \frac{\partial \psi}{\partial n} = 0, & x \in \Gamma, \\ \int_{\Omega} \psi(x) dx = 0 \end{cases}$$

and put

$$\|u\|_{-1}^2 = \int_{\Omega} |\nabla \psi|^2 dx + |\Omega| \bar{u}^2.$$

To apply theorem I.1.1 in [13], it is necessary to show

**Theorem 7.** For any  $\delta \geq 0$ , there exist absorbing sets in  $\mathcal{H}_\delta$  and in  $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$  for semigroup  $S(t)$  associated to problem (1.1).

**Proof of Theorem 7.** We first consider the existence of an absorbing set in  $\mathcal{H}_\delta$ . Multiplying the equation for  $u$  by  $\psi$  and the equation for  $v$  by  $v$  and integrating by parts respectively, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\bar{u}^2}{\alpha b^2} \|\nabla v\|^2 + \|u\|^2 \\ + (1 + \frac{2\bar{u}^2}{b^2}) \int_{\Omega} u^2 v^2 dx + \frac{3}{4} \bar{u}^2 \|v\|_{L^4}^4 \leq (2 + \frac{a^4}{b^4}) \bar{u}^2 |\Omega|. \end{aligned}$$

And we use the inequalities

$$\begin{aligned} \frac{\lambda_2}{2} \|u\|_{-1}^2 &\leq \frac{1}{2} \|u\|^2 + \frac{\lambda_2}{2} \bar{u}^2 |\Omega|, \\ \frac{\lambda_2}{2} \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2 &\leq \frac{\bar{u}^2}{4} \|v\|_{L^4}^4 + (\frac{\lambda_2}{\alpha b^2})^2 \bar{u}^2 |\Omega|, \end{aligned}$$

where  $\lambda_2$  is the least positive eigenvalue of the  $-\Delta$  with homogeneous Neumann boundary condition to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2) + \frac{\lambda_2}{2} (\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2) + \|\nabla u\|^2 + \frac{2\beta\bar{u}^2}{\alpha b^2} \|\nabla v\|^2 + \frac{1}{2} \|u\|^2 \\ + (1 + \frac{2\bar{u}^2}{b^2}) \int_{\Omega} u^2 v^2 dx + \frac{1}{2} \bar{u}^2 \|v\|_{L^4}^4 \leq \left(2 + \frac{a^4}{b^4} + \frac{\lambda_2}{2} + (\frac{\lambda_2}{\alpha b^2})^2\right) \bar{u}^2 |\Omega| \equiv \frac{C_1}{2}. \end{aligned}$$

And we have

$$(3.2) \quad \frac{d}{dt} (\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2) + \lambda_2 (\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2) \leq C_1.$$

Applying Gronwall's lemma to (3.2) we deduce for all

$$\|u\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v\|^2 \leq \left( \|u_0\|_{-1}^2 + \frac{2\bar{u}^2}{\alpha b^2} \|v_0\|^2 \right) e^{-\lambda_2 t} + \frac{C_1}{\lambda_2} (1 - e^{-\lambda_2 t}).$$

We have obtained an absorbing set in  $\mathcal{H}_\delta$ .

Next we show the existence of an absorbing set in  $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$ . Multiplying the equation for  $u$  by  $\Delta^2 u$  and integrating by parts yield

$$(3.3) \quad \frac{1}{2} \frac{d}{dt} \|\Delta u\|^2 + \frac{1}{2} \|\Delta^2 u\|^2 + 2 \|\nabla \Delta u\|^2 \leq \frac{1}{2} \|\Delta(uv^2)\|^2.$$

Multiplying the equation for  $v$  by  $\Delta^2 v$  and integrating by parts, we have

$$\begin{aligned} (3.4) \quad \frac{1}{2} \frac{d}{dt} \|\Delta v\|^2 + \beta \|\nabla \Delta u\|^2 &= -\alpha \int_{\Omega} \nabla[v(a^2 - u^2 - b^2 v^2)] \cdot \nabla \Delta v dx \\ &\leq \frac{\beta}{2} \|\nabla \Delta v\|^2 + \frac{3\alpha^2}{2\beta} (a^4 \|\nabla v\|^2 + \|\nabla(vu^2)\|^2 + 9b^4 \|v^2 \nabla v\|^2). \end{aligned}$$

Using Lemmas 5 and 6, the right hand side of (3.3) leads to

$$\begin{aligned} \|\Delta(uv^2)\|^2 &= \|v^2 \Delta u + 2uv \Delta v + 4v \nabla u \cdot \nabla v + 2u |\nabla v|^2\|^2 \\ &\leq C(\|v\|_{L^\infty}^4 \|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|v\|_{L^\infty}^2 \|\Delta v\|^2 + \|v\|_{L^\infty}^2 \|\nabla u\|_{L^4}^2 \|\nabla v\|_{L^4}^2 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4) \\ (3.5) \quad &\leq C(\|\Delta u\|^2 + \|u\|_{L^\infty}^2 \|\Delta v\|^2 + \|\nabla u\|_{L^4}^2 \|\nabla v\|_{L^4}^2 + \|u\|_{L^\infty}^2 \|\nabla v\|_{L^4}^4) \end{aligned}$$

and

$$\begin{aligned} (3.6) \quad \|\nabla(vu^2)\|^2 &= \|u^2 \nabla v + 2uv \nabla u\|^2 \\ &\leq C(\|\nabla v\|^2 \|u\|_{L^\infty}^4 + \|v\|_{L^\infty}^2 \|u\|_{L^\infty}^2 \|\nabla u\|^2) \\ &\leq C(\|u\|_{L^\infty}^4 + \|u\|_{L^\infty}^2). \end{aligned}$$

Using the inequalities (see, [13] p.161 and p.52)

$$\begin{aligned}\|\nabla v\|_{L^4} &\leq C\|\nabla v\|_{H^{\frac{3}{4}}} \leq C\|\nabla v\|^{\frac{1}{4}}\|\nabla^2 v\|^{\frac{3}{4}}, \\ \|u\|_{L^\infty} &\leq C\|u\|_{H^1}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}},\end{aligned}$$

we find, for example,

$$\begin{aligned}\|u\|_{L^\infty}^2\|\nabla v\|_{L^4}^4 &\leq C(\|u\|_{H^1}^{\frac{1}{2}}\|u\|_{H^2}^{\frac{1}{2}})^2(\|\nabla v\|^{\frac{1}{4}}\|\nabla^2 v\|^{\frac{3}{4}})^4 \\ &\leq C\|\Delta u\|\|\Delta v\|^3 + C' \\ &\leq C(\|\Delta u\|^4 + \|\Delta v\|^4) + C' .\end{aligned}$$

After some similar calculation, we obtain that

$$(3.7) \quad \frac{d}{dt}(\|\Delta u\|^2 + \beta\|\Delta v\|^2) \leq C_2(\|\Delta u\|^2 + \beta\|\Delta v\|^2)(\|\Delta u\|^2 + \beta\|\Delta v\|^2) + C_3.$$

Multiplying the equations for  $u$  by  $u$  and the equations for  $v$  by  $\Delta v$  and integrating by parts respectively, we have

$$(3.8) \quad \frac{d}{dt}(\|u\|^2 + \|\nabla v\|^2) + \|\Delta u\|^2 + \beta\|\Delta v\|^2 \leq C_4.$$

Here we have used Lemmas 5 and 6. By integrating (3.8), we find that the conditions of the uniform Gronwall lemma ([13] p.91) hold. Therefore we have

$$(3.9) \quad \|\Delta u(t)\|^2 + \beta\|\Delta v(t)\|^2 \leq (C_5 + C_4 + C_3)e^{C_2(C_5+C_4)}$$

for  $t \geq 1$ . From (3.9) we conclude the existence of an absorbing set in  $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$ .  $\square$

#### 4. THE INERTIAL SET

Let  $B_\delta$  be the absorbing set in  $(H^2(\Omega))^2 \cap \mathcal{H}_\delta$  from Theorem 7 and put  $X_\delta = \overline{\cup_{t \geq t_0} S(t)B_\delta}$ . We note that  $X_\delta$  is bounded in  $(C(\bar{\Omega}))^2$ .

**Lemma 8.** *The semigroup  $S(t) : X_\delta \rightarrow X_\delta$  is Lipschitz continuous, i.e.,*

$$(4.1) \quad \|(u_1 - u_2, v_1 - v_2)\|_H^2 \leq \|(u_{01} - u_{02}, v_{01} - v_{02})\|_H^2 e^{2dt},$$

where  $(u_i, v_i)$  is the solutions of (1.1) with the initial conditions  $(u_{0i}, v_{0i})$ ,  $i = 1, 2$ , and  $\|(u, v)\|_H^2 = \|u\|_{-1}^2 + \|v\|^2$ .

**Proof of Lemma 8.** The difference of solution  $(u_1 - u_2, v_1 - v_2)$  satisfies

$$(4.2) \quad \begin{cases} \frac{\partial(u_1 - u_2)}{\partial t} = \Delta(-\Delta(u_1 - u_2) + 2(u_1 - u_2) + u_1 v_1^2 - u_2 v_2^2), \\ \frac{\partial(v_1 - v_2)}{\partial t} = \beta\Delta(v_1 - v_2) + \alpha(v_1 - v_2)(a^2 - u_1^2 - b^2 v_1^2) - \alpha v_2 \{(u_1^2 - u_2^2) + b^2(v_1^2 - v_2^2)\}, \\ \frac{\partial(u_1 - u_2)}{\partial n} = \frac{\partial\Delta(u_1 - u_2)}{\partial n} = 0, \quad \frac{\partial(v_1 - v_2)}{\partial n} = 0, \quad (x, t) \in \Gamma_T, \\ u_1(x, 0) - u_2(x, 0) = u_{01}(x) - u_{02}(x), \quad v_1(x, 0) - v_2(x, 0) = v_{01}(x) - v_{02}(x), \quad x \in \Omega, \end{cases}$$

Let  $\psi_i$  be the solution of (3.1) with replacing  $u - \bar{u}$  by  $u_i - \bar{u}$ ,  $i = 1, 2$ . Multiplying the first equation of (4.2) by  $\psi = \psi_1 - \psi_2$  and multiplying the second equation of (4.2) by  $v_1 - v_2$  and

integrating by parts respectively, we get

$$(4.3) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(u_1 - u_2, v_1 - v_2)\|_H^2 + \|\nabla(u_1 - u_2)\|^2 + \|u_1 - u_2\|^2 + \beta \|\nabla(v_1 - v_2)\|^2 \\ & + \int_{\Omega} v_1^2 (u_1 - u_2)^2 dx + \alpha \int_{\Omega} (v_1 - v_2)^2 (u_1^2 + b^2 v_1^2) dx \leq d \|v_1 - v_2\|^2, \end{aligned}$$

where  $d = \alpha a^2 + 2\alpha b^2 M^2 + \frac{1}{4} M^2 (\alpha + 4)(1 + 4\alpha)$  and  $M$  is a constant such that  $\|(u_i, v_i)\|_{C(\bar{\Omega} \times [0, \infty))} \leq M$ . Applying Gronwall lemma leads to (4.1).  $\square$

Moreover we find from (4.3)

**Corollary 9.**

$$(4.4) \quad \int_0^t \|u_1 - u_2\|_{H^1}^2 e^{Ds} ds \leq \frac{1}{2} (1 + e^{(D+2d)t}) \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2,$$

where  $D$  is a constant.

**Proof of Corollary 9.** From (4.3), we have

$$(4.5) \quad \frac{1}{2} \frac{d}{dt} \|(u_1 - u_2, v_1 - v_2)\|_H^2 + \|u_1 - u_2\|_{H^1}^2 \leq d \|v_1 - v_2\|.$$

Multiplying (4.5) by  $e^{Ds}$  and integrating, we have

$$(4.6) \quad \int_0^t \|u_1 - u_2\|_{H^1}^2 e^{Ds} ds \leq \frac{1}{2} \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2 + \left(\frac{1}{2} D + d\right) \int_0^t \|(u_1 - u_2, v_1 - v_2)\|_H^2 e^{Ds} ds.$$

By using (4.1),

$$(4.7) \quad \begin{aligned} \int_0^t \|u_1 - u_2\|_{H^1}^2 e^{Ds} ds & \leq \frac{1}{2} \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2 \\ & + \left(\frac{1}{2} D + d\right) \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2 \int_0^t e^{(D+2d)s} ds \\ & \leq \frac{1}{2} (1 + e^{(D+2d)t}) \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2. \end{aligned}$$

$\square$

Next we shall show the squeezing property of  $S(t)$ . We denote by  $\lambda_i$ , ( $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_i \leq \dots$ ) the eigenvalue of the operator  $-\Delta$  with homogeneous Neumann boundary conditions and  $w_i$  the corresponding eigenfunctions such that  $\|w_i\|_{L^2} = 1, i = 1, 2, \dots$ . It is well-known that  $\{w_i\}_{i=1}^{\infty}$  are a complete orthogonal basis in  $L^2(\Omega)$ . Let  $H_n = \text{span}\{w_1, \dots, w_n\}$  and the operator  $p_n : (H^1(\Omega))' \rightarrow H_n$  be orthogonal projection and  $q_n = I - p_n$ , where  $I$  is identity on  $(H^1(\Omega))'$ . Then it holds that

$$(4.8) \quad \|\varphi\|_{-1}^2 \leq \frac{1}{\lambda_{n+1}} \|\varphi\|^2 \leq \frac{1}{\lambda_{n+1}^2} \|\nabla \varphi\|^2$$

for any  $\varphi \in q_n((H^1(\Omega))')$ . Furthermore we define the corresponding product projection  $P_n(u, v)$ ,  $Q_n(u, v)$  on  $H$  such that  $P_n(u, v) = (p_n u, p_n v)$ ,  $Q_n = I - P_n$ .

**Theorem 10.** Semigroup  $S(t)$  for problem (1.1) possesses the squeezing property, i.e., for any  $t^* > 0$  there exists number  $n_0 = n_0(t^*)$  such that for any  $\Psi_1 = (u_1, v_1), \Psi_2 = (u_2, v_2) \in X_\delta$  satisfying that if

$$(4.9) \quad \|p_{n_0}(S(t^*)\Psi_1 - S(t^*)\Psi_2)\|_H \leq \|(I - p_{n_0})(S(t^*)\Psi_1 - S(t^*)\Psi_2)\|_H,$$

then

$$(4.10) \quad \|S(t^*)\Psi_1 - S(t^*)\Psi_2\|_H \leq \frac{1}{8} \|\Psi_1 - \Psi_2\|_H.$$

**Proof of Theorem 10.** We set  $(U, V) \equiv Q_n(u_1 - u_2, v_1 - v_2)$ . Operating the equation (4.2) by  $Q_n$ , it holds that

$$(4.11) \quad \begin{cases} \frac{\partial U}{\partial t} = \Delta(-\Delta U + 2U + q_n(u_1 v_1^2 - u_2 v_2^2)), \\ \frac{\partial V}{\partial t} = \beta \Delta V + \alpha a^2 V - \alpha q_n(v_1(u_1^2 + b^2 v_1^2) - v_2(u_2^2 + b^2 v_2^2)). \end{cases}$$

Multiplying the first equation of (4.11) by  $NU \in q_n(H^2(\Omega))$  and the second equation of (4.11) by  $V$  and integrating, we have

$$(4.12) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \|(U, V)\|_H^2 + \|\nabla U\|^2 + 2\|U\|^2 + \beta \|\nabla V\|^2 &\leq \|U\|^2 + \alpha(a^2 + 1)\|V\|^2 + \frac{1}{4} \int_{\Omega} (u_1 v_1^2 - u_2 v_2^2)^2 dx \\ &\quad + \frac{\alpha}{4} \int_{\Omega} [v_1(u_1^2 + b^2 v_1^2) - v_2(u_2^2 + b^2 v_2^2)]^2 dx. \end{aligned}$$

Using the inequalities

$$(4.13) \quad \begin{cases} (\lambda_{n+1}^2 + \lambda_{n+1})\|U\|_{-1}^2 \leq \|\nabla U\|^2 + \|U\|^2, \\ \beta \lambda_{n+1} \|V\|^2 \leq \beta \|\nabla V\|^2, \end{cases}$$

it yields that

$$(4.14) \quad \frac{d}{dt} \|(U, V)\|_H^2 + (D_1 \lambda_{n+1} - D_2) \|(U, V)\|_H^2 \leq C(\|u_1 - u_2\|^2 + \|v_1 - v_2\|^2),$$

where  $D_1 = 2 \min\{1, \beta\}$ ,  $D_2 = 2\alpha(a^2 + 1)$ . Applying Gronwall's lemma leads to

$$(4.15) \quad \begin{aligned} \|(U, V)\|_H^2 &\leq \|(U, V)(0)\|_H^2 e^{-Dt} + \varepsilon e^{-Dt} \int_0^t \|u_1 - u_2\|_{H^1}^2 e^{Ds} ds \\ &\quad + C_{\varepsilon} e^{-Dt} \int_0^t (\|u_1 - u_2\|_{-1}^2 + \|v_1 - v_2\|^2) e^{Ds} ds, \end{aligned}$$

where  $D = D_1 \lambda_{n+1} - D_2$ . Here we use the inequality  $\|u_1 - u_2\|^2 \leq C\|u_1 - u_2\|_{-1}\|u_1 - u_2\|_{H^1}$ . From Lemma 8 and Corollary 9, it holds that

$$(4.16) \quad \|(U, V)(t)\|_H^2 \leq \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2 \left\{ e^{-Dt} + \frac{\varepsilon}{2}(e^{-Dt} + e^{2dt}) + C_{\varepsilon} \frac{e^{2dt}}{D + 2d} \right\}.$$

Now assume that

$$(4.17) \quad \|p_{n_0}(u_1 - u_2, v_1 - v_2)(t^*)\|_H \leq \|Q_{n_0}(u_1 - u_2, v_1 - v_2)(t^*)\|_H$$

for  $t^* > 0$ , then by using (4.16)

$$(4.18) \quad \begin{aligned} \|(u_1 - u_2, v_1 - v_2)(t^*)\|_H^2 &\leq 2\|Q_{n_0}(u_1 - u_2, v_1 - v_2)(t^*)\|_H^2 \\ &\leq \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2 \left\{ e^{-D_3 t^*} + \frac{\varepsilon}{2}(e^{-D_3 t^*} + e^{2dt^*}) + C_{\varepsilon} \frac{e^{2dt^*}}{D_3 + 2d} \right\}, \end{aligned}$$

where  $D_3 = D_1 \lambda_{n_0+1} - D_2$ . Taking  $\varepsilon > 0$  so small that

$$(4.19) \quad \varepsilon(e^{-D_3 t^*} + e^{2dt^*}) \leq \frac{1}{128}$$

and choosing a number  $n_0$  sufficiently large so as to satisfy

$$(4.20) \quad 2(e^{-D_3 t^*} + \frac{C_{\varepsilon} e^{2dt^*}}{D_3 + 2d}) \leq \frac{1}{128},$$

then we conclude

$$(4.21) \quad \|(u_1 - u_2, v_1 - v_2)(t^*)\|_H^2 \leq \left(\frac{1}{8}\right)^2 \|(u_{10} - u_{20}, v_{10} - v_{20})\|_H^2.$$

□

Therefore the proof of Theorem 2 is completed if we apply Theorem 2.1 in [6].

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